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ON SYMMETRIC FUNCTIONS.

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[Continued from February Number.]

c. With the first four equations, we have disposed of 16 of the 20 functions. Only four of them needed calculating. They were :

$$b_0^3\Sigma\beta_1^3\beta_2^3,\ b_0^2\Sigma\beta_1^2\beta_2^2,\ b_0^3\Sigma\beta_1^3,\ b_0^2\Sigma\beta_1^2.$$

The remaining 12 were the six functions already mentioned under a, and the following six :

$$|0^21|=b_0^3\Sigma\beta_1^3\beta_2^3\beta_3^2=b_0^2(\beta_1\beta_2\beta_3)^2b_0\Sigma\beta_1\beta_2,$$

$$|0^22|=b_0^3\Sigma\beta_1^3\beta_2^3\beta_3=b_0(\beta_1\beta_2\beta_3)b_0^2\Sigma\beta_1^2\beta_2^2,$$

$$|01^2|=b_0^3\Sigma\beta_1^3\beta_2^2\beta_3^2=b_0(\beta_1\beta_2\beta_3)^2b_0\Sigma\beta_1,$$

$$|1^22|=b_0^3\Sigma\beta_1^2\beta_2^2\beta_3=b_0b_0(\beta_1\beta_2\beta_3)b_0\Sigma\beta_1\beta_2,$$

$$|02^2|=b_0^3\Sigma\beta_1^3\beta_1\beta_2=b_0(\beta_1\beta_2\beta_3)b_0^2\Sigma\beta_1^2,$$

$$|12^2|=b_0^3\Sigma\beta_1^2\beta_2\beta_3=b_0b_0(\beta_1\beta_2\beta_3)b_0\Sigma\beta_1.$$

d. The remaining four functions may be obtained from four of the last six equations. The functions are :

$$|\,012\,| = b_0^3 \Sigma \beta_1^3 \beta_2^2 \beta_3 = b_0 (\beta_1 \beta_2 \beta_3) b_0^2 \Sigma \beta_1^2 \beta_2,$$

$$|\,013\,| = b_0^3 \Sigma \beta_1^3 \beta_2^2,$$

$$|\,023\,| = b_0^3 \Sigma \beta_1^3 \beta_2,$$

$$|\,123\,| = b_0^3 \Sigma \beta_1^2 \beta_2.$$

On account of the decomposition of the first into a known function and one of the other three, only three of these four require actual calculation.

3. CONTINUATION OF THE METHOD.

This method may be continued for higher resultants, what was said under (3) regarding the analysis of the operator δ of Aronhold into three others being capable of the easiest extension to the general case in that in the statements there made one must substitute the numbers 0, 1, 2, $\ldots . n$, and the literal factors $b_0, b_1, \ldots\ldots b_n$ instead of 0, 1, 2, 3, and b_0, b_1, b_2, b_3, respectively, and then these operators must be applied to all the $\frac{(n+1)(n+2)\ldots\ldots(2n-1)}{(n-1)!}$ stroked forms of $(n+1)$ elements to $(n-1)$ dimensions, thus obtaining $\frac{(n+1)(n+2)\ldots.(2n-1)}{(n-1)!}$ identical equations from which the $\frac{(n+1)(n+2)\ldots.2n}{n\,!}$ symmetric functions in connection with the $2n$ which may be already assumed as known, can be found. We have thus at the same time a method for elimination by means of symmetric functions, and a method for computing the values of the symmetric functions, giving the function as a whole. The functions for intermediate forms where m and n are unequal $(m>n)$ are contained in those forms where $n=m$.

II. ISOLATION OF TERMS OF A SYMMETRIC FUNCTION.

A. PRELIMINARY STATEMENTS CONCERNING THE RESULTANT THEORY.

1. THE CONCEPT OF NORMAL AND REDUCIBLE FORMS.

(1). *Definitions.*

In the thesis of the writer entitled "Die Entwickelung der Sylvester'schen Determinante nach Normal-Formen," (B. G. Teubner, Leipzig, 1898) the forms

$$P_{m,n} = a_{\kappa_1} a_{\kappa_2} \ldots . a_{\kappa_n} b_{\lambda_1} b_{\lambda_2} \ldots b_{\lambda_m} = (a_{r_1})^{p_1} (a_{r_2})^{p_2} \ldots . (a_{r_\mu})^{p_\mu} (b_{s_1})^{q_1} (b_{s_2})^{q_2} \ldots . (b_{s_\nu})^{q_\nu},$$

which occur in the resultant $R_{m,n}$ of

$$f = a_0 x^m - a_1 x^{m-1} + \ldots . (-1)^m a_m \text{ and } \phi = b_0 x^n + b_1 x^{n-1} + \ldots . + b_n$$

are divided into such as have the four factors a_0, b_0, a_m, b_n, and those which do

not have all four factors.* The former are called normal forms. It is shown that the latter are capable of being reduced by means of one or more of four kinds of reduction to normal forms of resultants $R_{\mu,\nu}$ of lower order and are called reducible forms, or else are *completely* reducible to a_0 or b_0 (in reality to unity), and are then called completely reducible forms.

(2). *Formulas.*

The normal forms have according to (1) the formula,

$$N_{m,n}=a_0 a_{\kappa_2}\ldots a_{\kappa_{n-1}} a_m b_0 b_{\lambda_2}\ldots b_{\lambda_{m-1}} b_n=(a_0)^{p_1}(a_{r_2})^{p_2}$$
$$\ldots(a_{r_{\mu-1}})^{p_{\mu-1}}(a_m)^{p_n}(b_0)^{q_1}(b_{s_2})^{q_2}\ldots(b_{s_{\nu-1}})^{q_{\nu-1}}(b_n)^{q_\nu}.$$

The normal form $N_{\mu,\nu}$ of the resultant $R_{\mu,\nu}$, to which a reducible term

$$P_{m,n}=(a_{r_1})^{p_1}(a_{r_2})^{p_2}\ldots(a_{r_\mu})^{p_\mu}(b_{s_1})^{q_1}(b_{s_2})^{q_2}\ldots(b_{s_\nu})^{q_\nu}$$

of $R_{m,n}$ is reduced, (by means of reductions to be stated in 2) has the formula

$$N_{\mu,\nu}=(a_{r_\lambda-\rho})^{p_\lambda-\alpha}(a_{r_{\lambda+1}-\rho})^{p_{\lambda+1}}\ldots(a_{r_\tau-\rho})^{p_\tau-r}(b_{s_\kappa-\sigma})^{q_\kappa-\beta}(b_{s_{\kappa+1}-\sigma})^{q_{\kappa+1}}\ldots(b_{s_\pi-\sigma})^{q_\pi-\delta},$$

where $r_\lambda=\rho$, $r_\tau=\mu+\rho$, $s_\kappa=\sigma$, $s_\pi=\nu+\sigma$, and where at least one of the numbers α, β, and at least one of the numbers r, δ is zero. Similarly, but without altering its value, the coefficient

$$C_{m,n}=(r_1)^{p_1}(r_2)^{p_2}\ldots(r_\mu)^{p_\mu}\mid(s_1)^{q_1}(s_2)^{q_2}\ldots(s_\nu)^{q_\nu}$$
$$\text{of } P_{m,n}=(a_{r_1})^{p_1}(a_{r_2})^{p_2}\ldots(a_{r_\mu})^{p_\mu}(b_{s_1})^{q_1}(b_{s_2})^{q_2}\ldots b_{s_\nu})^{q_\nu}$$

is reduced in form to $C_{\mu,\nu}$, so that $C_{m,n}=C_{\mu,\nu}$, or

$$(r_1)^{p_1}(r_2)^{p_2}\ldots(r_\mu)^{p_\mu}\mid(s_1)^{q_1}(s_2)^{q_2}\ldots s_\nu)^{q_\nu}$$
$$=(r_\lambda-\rho)^{p_\lambda-\alpha}(r_{\lambda+1}-\rho)^{p_{\lambda+1}}\ldots(r_\tau-\rho)^{p_\tau-r}\mid(s_\kappa-\sigma)^{q_\kappa-\beta}(s_{\kappa+1}-\sigma)^{q_{\kappa+1}}\ldots(s_\pi-\sigma)^{q_\pi-\delta}$$

The completely reducible forms of $R_{m,n}$ have the general formula :

$$v_{m,n}=(a_{r_1})^{p_1}(a_{r_2})^{p_2}\ldots(a_{r_\mu})^{p_\mu}(b_0)^{r_1}(b_{p_1})^{r_2-r_1}(b_{p_1+p_2})^{r_3-r_2}\ldots(b_{p_1+p_2+\ldots p_\mu})^{m-r_\mu}.$$

2. THE FOUR KINDS OF REDUCTION.

The before-mentioned reductions are attained in the following ways :

(1). *The first reduction.*

By dividing $P_{m,n}=(a_{r_1})^{p_1}(a_{r_2})^{p_2}\ldots(a_{r_\mu})^{p_\mu}(b_{s_1})^{q_1}(b_{s_2})^{q_2}\ldots(b_n)^{q_\nu}$, a form of $R_{m,n}$, which contains the factor b_n, but not a_m, by $(b_n)^{m-r_\mu}$, $P_{m,n}$ is reduced to $P_{r_\mu,n}=(a_{r_1})^{p_1}(a_{r_2})^{p_2}\ldots(a_{r_\mu})^{p_\mu}(b_{s_1})^{q_1}(b_{s_2})^{q_2}\ldots b_n{}^{q_\nu-m+r_\nu}$, a form of a lower resultant $R_{r_\mu,n}$.

*It is a fundamental condition that every form $P_{m,n}$ must contain at least one of the factors a_0, b_0, and at least one of the factors a_m, b_n.

(2). *The second reduction.*

By dividing $P_{m,n}=(a_{r_1})^{p_1}(a_{r_2})^{p_2}\ldots(a_m)^{p_\mu}(b_{s_1})^{q_1}(b_{s_2})^{q_2}\ldots(b_{s_\nu})^{q_\nu}$ of $R_{m,n}$, which has the factor a_m but not b_n, by $(a_m)^{n-s_\nu}$, we obtain $P_{m,s_\nu}=(a_{r_1})^{p_1}(a_{r_2})^{p_2}\ldots(a_m)^{p_\mu-n+s_\nu}(b_{s_1})^{q_1}(b_{s_2})^{q_2}\ldots(b_{s_\nu})^{q_\nu}$, a form of the resultant R_{m,s_ν} of lower order than $R_{m,n}$.

(3). *The third reduction.*

Here $P_{m,n}=(a_{r_1})^{p_1}(a_{r_2})^{p_2}\ldots(a_{r_\mu})^{p_\mu}(b_0)^{q_1}(b_{s_2})^{q_2}\ldots(b_{s_\nu})^{q_\nu}$ (with $r_1 \neq 0$) is reduced to $P_{m-r_1,n}=(a_0)^{p_1}(a_{r_2-r_1})^{p_2}\ldots(a_{r_\mu-r_1})^{p_\mu}(b_0)^{q_1-r_1}(b_{s_2})^{q_2}\ldots(b_{s_\nu})^{q_\nu}$, a form of the resultant $R_{m-r_1,n}$, by dividing by $(b_0)^{r_1}$ and diminishing the subscripts of the a's by r_1.

(4). *The fourth reduction.*

In this case $P_{m,n}=(a_0)^{p_1}(a_{r_2})^{p_2}\ldots\ldots(a_{r_\mu})^{p_\mu}(b_{s_1})^{q_1}(b_{s_2})^{q_2}\ldots\ldots(b_{s_\nu})^{q_\nu}$, $(s_1 \neq 0)$, is reduced to $P_{m,n-s_1}=(a_0)^{p_1-s_1}(a_{r_2})^{p_2}\ldots(a_{r_\mu})^{p_\mu}(b_0)^{q_1}(b_{s_2-s_1})^{q_2}\ldots(b_{s_\nu-s_1})^{q_\nu}$, of the resultant $R_{m,n-s_1}$, by division by $(a_0)^{s_1}$, and diminution of the subscripts of the b's by s_1.

3. THE FOUR KINDS OF DERIVATION.

Conversely we may start with forms of lower resultants and by four kinds of derivation attain to forms of higher resultants.

(1). *The first kind of derivation.*

By multiplying $P_{m,n}=(a_{r_1})^{p_1}(a_{r_2})^{p_2}\ldots(a_{r_\mu})^{p_\mu}(b_{s_1})^{q_1}(b_{s_2})^{q_2}\ldots(b_{s_\nu})^{q_\nu}$ by $b_n{}^q$, we obtain $P_{m+q,n}=(a_{r_1})^{p_1}(a_{r_2})^{p_2}\ldots(a_{r_\mu})^{p_\mu}(b_{s_1})^{q_1}(b_{s_2})^{q_2}\ldots(b_{s_\nu})^{q_\nu}b_n{}^q$, a form of the resultant $R_{m+q,n}$.

(2). *The second kind of derivation.*

Similarly from $P_{m,n}$ we obtain $P_{m,n+p}=(a_{r_1})^{p_1}(a_{r_2})^{p_2}\ldots(a_{r_\mu})^{p_\mu}(a_m)^p(b_{s_1})^{q_1}(b_{s_2})^{q_2}\ldots(b_{s_\nu})^{q_\nu}$, of the resultant $R_{m,n+p}$, by multiplication by $a_m{}^p$.

(3). *The third kind of derivation.*

In this case we obtain $P_{m+q,n}=(a_{r_1+q})^{p_1}(a_{r_2+q})^{p_2}\ldots(a_{r_\mu+q})^{p_\mu}(b_0)^q(b_{s_1})^{q_1}\ldots(b_{s_\nu})^{q_\nu}$, of the resultant $R_{m+q,n}$, from $P_{m,n}$ by multiplication by $b_0{}^q$ and increase of the subscripts of the a's by q.

(4). *The fourth kind of derivation.*

Here we multiply $P_{m,n}$ by $a_0{}^p$ and increase the subscripts of the b's by p, and obtain $P_{m,n+p}=(a_0)^p(a_{r_1})^{p_1}\ldots(a_{r_\mu})^{p_\mu}(b_{s_1+p})^{q_1}(b_{s_2+p})^{q_2}\ldots(b_{s_\nu+p})^{q_\nu}$, of the resultant $R_{m,n+p}$.

4. RECURRENCE FORMULA FOR THE NORMAL COEFFICIENTS.

For the calculation of the coefficients of the normal forms we use the formula

$$p_1 \times 0^{p_1}(r_2)^{p_2}\ldots m^{p_\mu} \mid 0^{q_1}(s_2)^{q_2}\ldots n^{q_\nu} = \sum_{\lambda=2}^{\lambda=\nu}(-1)^{s_\lambda+1}(c_\lambda+1)$$

$$\times s_\lambda\, 0^{p_1-1}(r_2)^{p_2}\ldots m^{p_\mu} \mid (s_\lambda)^{-1}0^{q_1+1}(s_2)^{q_2}\ldots n^{q_\nu},$$

where c_λ is the exponent of s_λ in the expression $0^{p_1-1}(r_2)^{p_2}\ldots m^{p_\mu}$. It is a recurrence formula and serves for the calculation of the coefficient of the normal form $(a_0)^{p_1}(a_{r_2})^{p_2}\ldots(a_m)^{p_\mu}(b_0)^{q_1}(b_{s_2})^{q_2}\ldots(b_n)^{q_\nu}$ from earlier calculated coeffi-

cients of simpler forms. In it every coefficient on the right hand is to be reduced to its simplest form according to 1, (2).

5. THE RESULTANT $R_{m,n}$ IN TERMS OF SYMMETRIC FUNCTIONS.

(1). *The expressions.*

The resultant $R_{m,n}$ of f and ϕ may be written in the forms,

$$(-1)^{mn}b_0{}^m f(\beta_1)f(\beta_2)\ldots f(\beta_n)=$$

$$(-1)^{mn}b_0{}^n[a_1\beta_1{}^m-a_1\beta_1{}^{m-1}+\ldots+(-1)^m a_m][a_0\beta_2{}^m-a_1\beta_1{}^{m-1}+\ldots+(-1)^m a_m]$$

$$\ldots[a_0\beta_n{}^m-a_1\beta_n{}^{m-1}+\ldots+(-1)^m a_m]=a_0{}^n\phi(\alpha_1)\phi(\alpha_2)\ldots\phi(\alpha_n)$$

$$=a_0{}^n(b_0\alpha_1{}^n+\ldots+b_n)(b_0\alpha_2{}^n+\ldots+b_n)\ldots(b_0\alpha_m{}^n+\ldots+b_n).$$

(2). *The coefficient* $(m-\varkappa_1)(m-\varkappa_2)\ldots(m-\varkappa_n)\mid 0^{\lambda_0}1^{\lambda_1}\ldots n^{\lambda_n})$.

The coefficient of $a_{m-\kappa_1}a_{m-\kappa_2}\ldots a_{m-\kappa_n}$, using the first form of $R_{m,n}$, is $(-1)^{\kappa_1+\kappa_2+\ldots+\kappa_n}b_0{}^m\Sigma\beta_1{}^{\kappa_1}\beta_2{}^{\kappa_2}\ldots\beta_n{}^{\kappa_n}$. In this again the coefficient $(m-\varkappa_1)(m-\varkappa_2)\ldots(m-\varkappa_n)\mid 0^{\lambda_0}1^{\lambda_1}\ldots n^{\lambda_n}$ of the product $a_{m-\kappa_1}a_{m-\kappa_2}\ldots a_{m-\kappa_n}b_0{}^{\lambda_0}b_1{}^{\lambda_1}\ldots b_n{}^{\lambda_n}$ in $R_{m,n}$, is equal numerically to the coefficient of $b_0{}^{\lambda_0}b_1{}^{\lambda_1}\ldots b_n{}^{\lambda_n}$ in $b_0{}^n\Sigma\beta_1{}^{\kappa_1}\beta_2{}^{\kappa_2}\ldots\beta_n{}^{\kappa_n}$, that is,

$$(m-\varkappa_1)(m-\varkappa_2)\ldots(m-\varkappa_n)\mid 0^{\lambda_0}1^{\lambda_1}\ldots n^{\lambda_n}=(-1)^{\kappa_1+\kappa_2+\ldots+\kappa_n}$$

$$\begin{pmatrix}0^{\lambda_0}1^{\lambda_1}\ldots\ldots\ldots\ldots n^{\lambda_n}\\ b_0{}^m\beta_1{}^{\kappa_1}\beta_2{}^{\kappa_2}\ldots\ \beta_n{}^{\kappa_n}\end{pmatrix},\ \text{where}\ \begin{pmatrix}0^{\lambda_0}1^{\lambda_1}\ldots\ldots\ldots\ldots n^{\lambda_n}\\ b_0{}^m\beta_1{}^{\kappa_1}\beta_2{}^{\kappa_2}\ldots\ \beta_n{}^{\kappa_n}\end{pmatrix}$$

denotes the coefficient of $b_0{}^{\lambda_0}b_1{}^{\lambda_1}\ldots b_n{}^{\lambda_n}$ in $b_0{}^m\Sigma\beta_1{}^{\kappa_1}\beta_2{}^{\kappa_2}\ldots\beta_n{}^{\kappa_n}$.

Using the second form of the resultant, the coefficient of $b_0{}^{\lambda_0}b_1{}^{\lambda_1}\ldots b_n{}^{\lambda_n}$ is

$$a_0{}^n\Sigma(\alpha_1\alpha_2\ldots\alpha_{\lambda_0})^n(\alpha_{\lambda_0+1}\ldots\alpha_{\lambda_0+\lambda_1})^{n-1}\ldots(\alpha_{\lambda_0+\lambda_1+\ldots\lambda_{n-1}}\ldots\alpha_{\lambda_0+\ldots\lambda_n})^0.$$

This last symmetric function we will express symbolically by

$$a_0{}^n\Sigma(\alpha\lambda_0)^n(\alpha\lambda_1)^{n-1}(\alpha\lambda_2)^{n-2}\ldots(\alpha\lambda_n)^0,$$

where $(\alpha\lambda_r)^{n-r}$ means that λ_r roots (α) have the exponent $n-r$.

In this symmetric function the coefficient

$$(m-\varkappa_1)(m-\varkappa_2)\ldots\ldots(m-\varkappa_n) \mid 0^{\lambda_0}1^{\lambda_1}\cdots\cdots n^{\lambda_n}$$

of the product

$$a_{m-\kappa_1}a_{m-\kappa_2}\ldots\ldots a_{m-\kappa_n}b_0{}^{\lambda_0}b_1{}^{\lambda_1}\ldots\ldots b_n{}^{\lambda_n}$$

in $R_{m,n}$ is equal numerically to the coefficient of $a_{m-\kappa_1}a_{m-\kappa_2}\ldots\ldots a_{m-\kappa_n}$ in $a_0{}^n\Sigma(\alpha\lambda_0)^n(\alpha\lambda_1)^{n-1}\ldots\ldots(\alpha\lambda_n)^0$, that is

$$(m-\varkappa_1)(m-\varkappa_2)\ldots\ldots(m-\varkappa_n) \mid 0^{\lambda_0}1^{\lambda_1}\ldots\ldots n^{\lambda_n}=(-1)^{mn+\kappa_1+\kappa_2+\ldots\kappa_n}$$

$$\begin{pmatrix}(m-\varkappa_1)(m-\varkappa_2)\ldots\ldots(m-\varkappa_n) \\ a_0{}^n(\alpha\lambda_0)^n(\alpha\lambda_1)^{n-1}\ldots\ldots(\alpha\lambda_n)^0\end{pmatrix}.$$

(3). *Relation between coefficients of terms in symmetric functions.*

By (2) we have two expressions for the coefficient of a term of $R_{m,n}$. By equating these expressions, we obtain a relation between the coefficients of terms of two symmetric functions, namely :

$$\begin{pmatrix}0^{\lambda_0}1^{\lambda_1}\ldots\ldots\ldots\ldots n^{\lambda_n} \\ b_0{}^m\beta_1{}^{\kappa_1}\beta_2{}^{\kappa_2}\ldots\ \ .\ \beta_n{}^{\kappa_n}\end{pmatrix}=(-1)^{mn}\begin{pmatrix}(m-\varkappa_1)(m-\varkappa_2)\ldots\ldots(m-\varkappa_n) \\ a_0{}^n(\alpha\lambda_0)^n(\alpha\lambda_1)^{n-1}\ldots\ldots(\alpha\lambda_n)^0\end{pmatrix},$$

which is seen to be true whether f is written with alternating sign or not.

With this paragraph the statements concerning the resultant theory, so far as they relate to this paper are finished. It is now proposed to develop a theory for symmetric functious similar to that sketched for the resultant, and to point out the correspondences between them. Moreover, the theory is developed independently of the results for the resultant, with the exception of the last formula, which will be used.

[To be continued.]